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Technical Note

Exact dual solutions occurring in the Darcy mixed convection flow

E. Magyari [a], I. Pop [b,*], B. Keller [a]

[a] Chair of Physics of Buildings, Institute of Building Technology, Swiss Federal Institute of Technology, Zürich, Switzerland
[b] Faculty of Mathematics, University of Cluj, CP 253, R-3400 Cluj, Romania

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1. Introduction

Convective flow in fluid-saturated porous media has been studied quite extensively over the last several decades. Numerous authors cite a wide variety of practical applications of this phenomena that includes utilization of geothermal energy, fiber and granular insulations, design of packed bed reactors, underground disposal of nuclear waste materials, etc. Several others investigate the intricate nature of solution structure from a fundamental point of view in some particular situations. Comprehensive reviews on the subject can be found in the books by Nield and Bejan [1], Nakayama [2], Ingham and Pop [3] and Vafai [4].

In practical applications most of the convective flows in porous media are non-isothermal. The assumption of wall temperature distribution which permits similarity solutions is very often used. Thus, analytical solutions of the flow and heat transfer can be obtained and therefore fluid flow and heat transfer characteristics can be easily analyzed.

Nakayama and Koyama [5] treated by introducing a general transformation procedure one of the most fundamental case of convective flow in porous media, namely the problem of combined free and forced convection over a plane or axisymmetric body of arbitrary shape which is embedded in a fluid-saturated porous medium. The analysis shows that unlike in pure forced convection, similarity solutions in mixed convection flow regime are possible only when the external free-stream velocity varies everywhere in properties to the product of the streamwise component of the gravity force and the wall-ambient temperature difference. However, no closed form analytical solutions were reported by Nakayama and Koyama [5].

In this note, we shall consider the same physical model of Nakayama and Koyama [5], and identify a particular flow regime when the mixed convection parameter Ra_x/Pe_x is constant; Ra_x and Pe_x are the local Rayleigh and Péclet numbers for a porous medium. Closed form analytical solutions are obtained, which subsequently lead to multiple (dual) solutions. However, Nakayama and Koyama [5] have not examined the multiplicity features of these solutions. It is also worth mentioning that Merkin [6,7] has studied dual solutions for the mixed convection boundary-layer flow over a vertical isothermal surface embedded in a porous medium the mixed convection parameter being noted by α. It was shown that the boundary value problem has just one solution for $\alpha \leqslant 0$ and no solutions for $\alpha > \alpha_0$ (≈ 0.354), while for $0 < \alpha < \alpha_0$ there were two solutions, an upper solution F_u and a lower solution F_l with $0 < F_l''(0) < F_u''(0)$.

2. Basic equations and solutions

Consider a plane or axisymmetric body of arbitrary shape embedded in a fluid-saturated porous medium. The geometry and wall temperature of the heated body are specified by the functions $r(x)$ and $T_w(x)$, where x is the coordinate measured along the surface of the body from its lower stagnation point. It is also assumed that the tangential component of the acceleration due to gravity g_x is a function of the wall geometry $r(x)$. Under the boundary layer and Darcy–Boussinesq approximations the governing equations of the steady mixed convection flow past a plane or axisymmetric body of arbitrary shape were reduced by Nakayama and Koyama [5] to the following form:

* Corresponding author.
E-mail address: popi@math.ubbcluj.ro (I. Pop).

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Nomenclature

b	mixed convection parameter, Eq. (6)
f	non-dimensional stream function
g	acceleration due to gravity
g_x	tangential component of the acceleration due to gravity g
I	function related to the body geometry and its surface temperature distribution
n	function related to the surface temperature distribution
nI	lumped parameter, Eq. (4)
Pe_x	local Péclet number
r	function representing wall geometry
$r^* = 1$	for plane flow and $r^* = r$ for axisymmetric flow
Ra_x	local Rayleigh number
x	coordinate measured along the surface of the body

Greek symbols

η	similarity variable
θ	non-dimensional temperature
λ	exponent associated with the wall temperature distribution
ξ	transformed variable in the streamwise direction, Eq. (5)
ΔT_w	difference between the wall temperature and ambient temperature, Eq. (7)

Superscript

$'$	differentiation with respect to η

$$f' = 1 + (Ra_x/Pe_x)\theta, \tag{1}$$

$$\theta'' + \left(\frac{1}{2} - nI\right) f\theta' - nIf'\theta = xI\left(f'\frac{\partial\theta}{\partial x} - \theta'\frac{\partial f}{\partial x}\right) \tag{2}$$

along with the boundary conditions

$$f = 0, \quad \theta = 1 \ \text{on} \ \eta = 0, \qquad f' \to 1, \quad \theta \to 0 \ \text{as} \ \eta \to \infty, \tag{3}$$

where primes denote partial differentiation with respect to η. The product nI is called the lumped parameter and is defined as

$$nI = \frac{\mathrm{d}\ln \Delta T_w}{\mathrm{d}\ln \xi}\,\frac{\int \Delta T_w^3\,\mathrm{d}\xi}{\xi \Delta T_w^3}, \tag{4}$$

where the transformed variable ξ is given by

$$\xi = \int_0^x g_x r^{*2}\,\mathrm{d}x \tag{5}$$

with $r^* = 1$ for plane flow and $r^* = r(x)$ for axisymmetric flow, respectively.

We will show now that Eqs. (1) and (2) admit a closed form analytical solution. Thus, assuming that

$$\frac{Ra_x}{Pe_x} = \text{const.} \equiv b, \tag{6}$$

$$\Delta T_w \propto \xi^{\lambda}, \tag{7}$$

then the lumped parameter nI has the form

$$nI = \frac{\lambda}{1 + 3\lambda} \tag{8}$$

and Eqs. (1) and (2) reduce to ordinary differential equations. We assume now that $\lambda = -1$. Hence, Eqs. (1) and (2) can be reduced as

$$2f''' + f' - f'^2 = 0 \tag{9}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 1 + b, \qquad f' \to 1 \quad \text{as} \ \eta \to \infty. \tag{10}$$

Eq. (8) admits for $f'(\eta)$ two solutions which satisfy the boundary conditions (10) and are real for any given value of the parameter b in the range

$$-\frac{3}{2} \leqslant b < 0. \tag{11}$$

These solutions are

$$f'_{\pm}(\eta) = -\frac{1}{2} + \frac{3}{2}\tanh^2\left[\frac{\eta}{2\sqrt{2}} \pm \frac{1}{2}\ln\left(\frac{\sqrt{3} + \sqrt{3+2b}}{\sqrt{3} - \sqrt{3+2b}}\right)\right] \tag{12}$$

so that the reduced wall skin friction is given by

$$f''_{\pm}(0) = \mp b\sqrt{\frac{2b+3}{6}}. \tag{13}$$

Also the non-dimensional temperature field can be expressed as:

$$\theta_{\pm}(\eta) = \frac{f'_{\pm}(\eta) - 1}{b} = \frac{3}{2|b|}\cosh^{-2}\left[\frac{\eta}{2\sqrt{2}} \pm \frac{1}{2}\ln\left(\frac{\sqrt{3} + \sqrt{3+2b}}{\sqrt{3} - \sqrt{3+2b}}\right)\right] \tag{14}$$

and the corresponding wall heat flux has the expression:

$$\theta'_{\pm}(0) = \frac{1}{b} f''_{\pm}(0) = \mp\sqrt{\frac{2b+3}{6}}. \tag{15}$$

3. Results and discussion

The dual temperature profiles (14) and the corresponding dimensionless downstream velocity profiles (12) are presented in Figs. 1–6 for some values of the mixed convection parameter b in the range $-3/2 \leqslant b < 0$. Despite all these solutions are real in the whole range $-3/2 \leqslant b < 0$, it is seen that:

- If $-3/2 \leqslant b < -1$, both the dual velocity profiles $f'_{\pm}(\eta)$ posses regions of reversed flow (where $f'_{\pm}(\eta) < 0$).
- If $-1 \leqslant b < 0$ the profile $f'_{-}(\eta)$ continues to show reversed flow, while the profile $f'_{+}(\eta)$ becomes positive for any $\eta \geqslant 0$ (except for $b = -1$, where according to the boundary conditions (10) $f'_{+}(0) = 0$ holds).

One may also conclude that for $-3/2 \leqslant b < -1$ all the solutions $\theta_{\pm}(\eta)$ and $f'_{\pm}(\eta)$ although real, are non-physical and the only physically realizable solutions of the problem are $\theta_{+}(\eta)$ and $f'_{+}(\eta)$ for $-1 \leqslant b < 0$.

Further, Figs. 7 and 8 show the variation with b of the wall heat flux $\theta'_{\pm}(0)$ and skin friction $f''_{\pm}(0)$. For our purpose these figures provide sufficient details on the

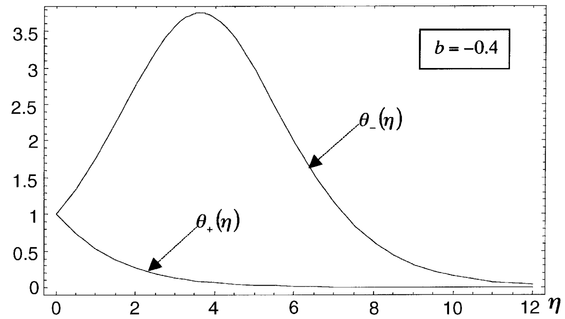

Fig. 1. Plot of $\theta_{\pm}(\eta)$ against η for $b = -0.4$.

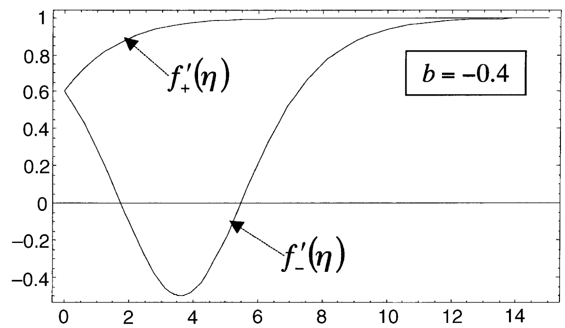

Fig. 2. Plot of $f'_{\pm}(\eta)$ against η for $b = -0.4$.

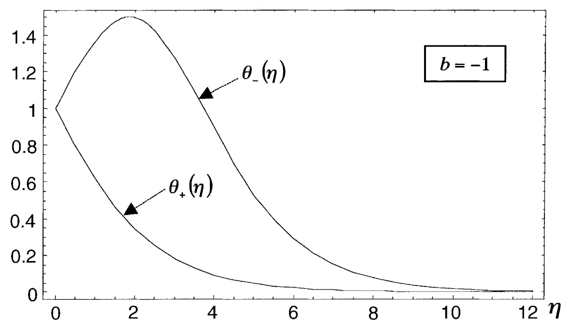

Fig. 3. Plot of $\theta_{\pm}(\eta)$ against η for $b = -1$.

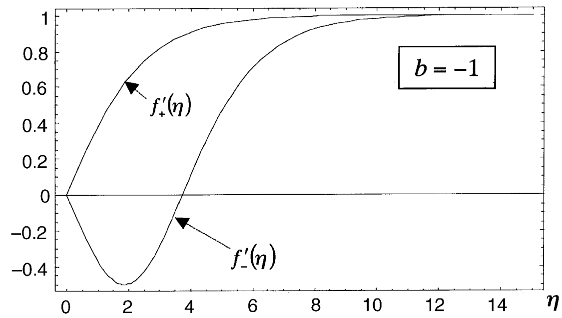

Fig. 4. Plot of $f'_{\pm}(\eta)$ against η for $b = -1$.

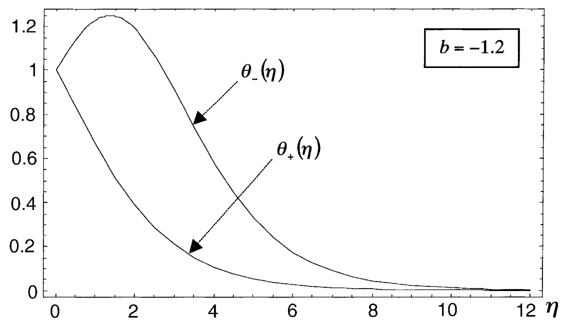

Fig. 5. Plot of $\theta_{\pm}(\eta)$ against η for $b = -1.2$.

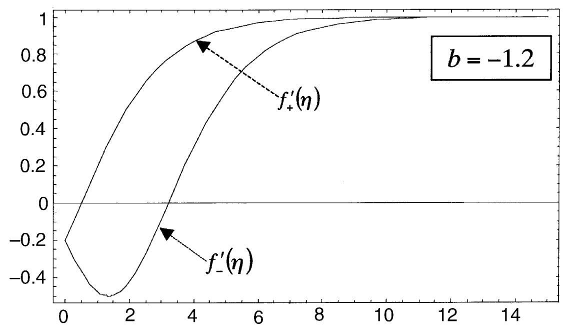

Fig. 6. Plot of $f'_{\pm}(\eta)$ against η for $b = -1.2$.

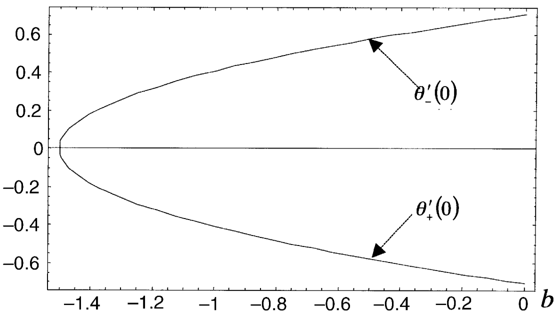

Fig. 7. Plot of $\theta'_{\pm}(0)$ against $b \in [-3/2, 0]$.

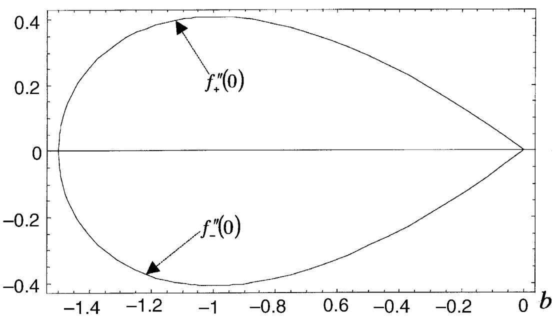

Fig. 8. Plot of $f''_{\pm}(0)$ against $b \in [-3/2, 0]$.

structure changes of the solution. It is clearly seen that both $\theta'_{\pm}(0)$ and $f''_{\pm}(0)$ are symmetric and that the symmetry breaking bifurcation of the solutions happens at the value $b = -3/2$. However, $f''_{+}(0)$ and $f''_{-}(0)$ connect with each other at $b = -3/2$ and $b \to 0$.

4. Conclusions

An exact analytical solution for the steady mixed convection boundary layer flow over a plane or axisymmetric body of arbitrary shape has been presented for the specific case where the wall temperature distribution varies according to the inverse linear distance in the streamwise direction. The multiplicity (dual) feature of the problem has been found for the mixed convection parameter $b = Ra_x/Pe_x$ in the range $-3/2 \leqslant b < -1$. These solutions, velocity and temperature profiles are such that $f'_{+}(\eta) \geqslant f'_{-}(\eta)$ and $\theta_{+}(\eta) \leqslant \theta_{-}(\eta)$ for a given value of $b \in [3/2, 0)$ and any $\eta \geqslant 0$ (the equality holds only for $\eta = 0$ and $\eta \to \infty$, respectively). It has been found that only the solutions $\theta_{+}(\eta)$ and $f'_{+}(\eta)$ are relevant to the physical problem and this is only when $-1 \leqslant b < 0$.

Finally, we mention that for $b > 0$ no multiple solutions are possible. In this range of the mixed convection parameter the solution of the boundary value problem (9)–(10) is unique for any $b > 0$. It is also available in a closed analytic form and reads:

$$f'(\eta) = 1 + \frac{3}{2\sinh^2\left[\frac{\eta}{2\sqrt{2}} + \ln\left(\sqrt{\frac{3}{2b}} + \sqrt{1 + \frac{3}{2b}}\right)\right]}. \qquad (16)$$

The corresponding dimensionless skin friction is given by

$$f''(0) = -b\sqrt{\frac{1}{2} + \frac{b}{3}}. \qquad (17)$$

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